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NON-TAME AUTOMORPHISMS OF EXTENSIONS OF PERIODIC GROUPS

BY

YOAV MORIAH* AND VLADIMIR SHPILRAIN**

Department of Mathematics
Technion—Israel Institute of Technology, Haifa 32000, Israel

ABSTRACT

Let F be a free group and $R \leq F$ a characteristic subgroup. Automorphisms of F/R which are induced by automorphisms of F are called **tame**. In this paper we use the $\mathcal{N}$-torsion invariant discovered by the first author and M. Lustig [LM] to show the existence of non-tame automorphisms of free central extensions and free nilpotent extensions of Burnside groups.

0. Introduction

Let $F = F_n$ be a free group of rank $n \geq 2$ on the set $X = \{X_1, ..., X_n\}$ of free generators. If R is a characteristic subgroup of F then the natural mapping σ_R: $F \to F/R$ induces the mapping θ_R: $\mathrm{Aut}F \to \mathrm{Aut}(F/R)$ of the corresponding automorphism groups. The automorphisms of the group F/R which belong to the image of θ_R are called **tame**. More generally, if R is an arbitrary normal subgroup of F, we call an automorphism φ of the group F/R **tame** if there exists an automorphism φ' of F such that $\varphi'(R) = R$ and φ' induces the automorphism φ of F/R in the obvious way.

The question of existence of non-tame automorphisms in general groups is stimulated by the classical result of Nielsen which describes the group $\mathrm{Aut}F$ in terms of generators and relations (see [N]). In particular, this group can be generated by four elements.

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Let $\gamma_c(G)$ denote the c-th term of the lower central series of a group G. Instead of $\gamma_2(G)$ we usually write G'. Bachmuth has proved that every free nilpotent group $F/\gamma_{c+1}(F)$ of class $c \geq 3$ and of rank $n \geq 2$ has non-tame automorphisms (see [B]). This result was previously partly proved by Andreadakis [A].

For free metabelian groups $M_n = F_n/F_n''$ it was proved by Bachmuth and Mochizuki that the groups M_2 and M_k, $k \geq 4$, have only tame automorphisms (see [BM1], [BM2]). However Chein showed that the group M_3 has non-tame ones (see [C]). This shows that, sometimes, the rank of the group plays a sensitive role in this question. It was later proved that $\mathrm{Aut}M_3$ is not even finitely generated (see [BM1]).

The second author has proved the existence of non-tame automorphisms in the rather general situation of groups of the form $F/\gamma_c(R)$, in particular, of free solvable non-metabelian groups of rank $n \geq 4$ (see [Sh1]). Similar results have been obtained by C.K. Gupta and Levin (see [GL]). Both approaches of [Sh1] and [GL] are based on finding necessary conditions for a matrix over a free group ring to be invertible. A rather strong and convenient necessary condition of that sort has been elaborated in [BGLM].

A different approach to the problem of finding non-tame automorphisms has been discovered recently by the first author and M. Lustig. They have found a new invariant for Nielsen equivalence classes of generating systems called $\mathcal{N}$-**Torsion**. Among other things, this invariant yields a necessary condition for an arbitrary automorphism of an arbitrary finitely generated group to be tame (see [LM]).

In this paper, we use the approach of [LM] to find non-tame automorphisms of various group extensions. First we demonstrate our method by reproving Bachmuth's result on automorphisms of free nilpotent groups cited above. Then we consider free central extensions of free Burnside groups $B(n,p) = F_n/F_n^p$ where F_n^p is the subgroup generated by all p-powers of elements of F, and $p \geq 2$ is an *arbitrary* integer. We prove the following:

THEOREM 0.1: *The group $F_n/[F_n^p, F_n]$ has non-tame automorphisms in the following cases:*

(a) *The exponent p is even; $p \geq 4$ and the rank $n \geq 3$;*
(b) *The exponent p is odd; $p \geq 3$ and the rank $n \geq 4$.*

Remark 0.2: The free Burnside groups are known to have non-tame automorphisms as can be seen from the following example:

Let F be the free group on $\{X_1, X_2\}$ and $p \geq 5$. Let q, r be two integers relatively prime to p so that $1 < qr < p-1$. Let x_i be the image of X_i under the natural epimorphism $F_2 \to B(2,p)$. It is clear that the map φ: $F/F^p \to F/F^p$ defined by $\varphi(x_1) = x_1^q$, $\varphi(x_2) = x_2^r$ defines an automorphism of F/F^p. If φ lifts to $\varphi' \in \mathrm{Aut} F$ then φ' will induce an automorphism of $Z \oplus Z$ given by a 2×2-matrix:

$$\begin{bmatrix} q+pj & pk \\ pl & r+pm \end{bmatrix} \quad \text{for some } j,k,l,m \in Z.$$

The determinant of this matrix is clearly not ± 1; hence a contradiction. ∎

We furthermore consider free nilpotent extensions of periodic groups and prove:

THEOREM 0.3: *Let $B(n,p)$ be the free Burnside group, then its free nilpotent extension $F_n/\gamma_c(F_n^p)$ has non-tame automorphisms for any rank $n \geq 2$ and $c \geq 3$.*

Remark 0.4: An endomorphism of F induces an automorphism of the group F/R' if and only if given any fully invariant subgroup V of F such that $V \leq R'$ and the quotient R/V is nilpotent, it induces an automorphism of the group F/V (see [BG] Lemma 3.1). ∎

COROLLARY 0.5: *The group $F/[\gamma_c(F_n^p), F]$ has non-tame automorphisms for any rank $n \geq 2$ and $c \geq 3$.*

Proof: For $c \geq 2$ the group $\gamma_c(R)/[\gamma_c(R), F_n]$ is the center of $F_n/[\gamma_c(R), F_n]$ (see [GG] and [Sh2]), hence it is a characteristic subgroup, so every automorphism of $F_n/[\gamma_c(F_n^p), F_n]$ induces an automorphism of $F_n/\gamma_c(F_n^p)$. Any non-tame automorphism φ of the group $F_n/\gamma_c(F_n^p)$, which exists by Theorem 0.3, induces some automorphism ψ of the group $F_n/[\gamma_c(F_n^p), F_n]$. This is an application of the cited Lemma 3.1 of [BG] for $R = F_n^p$ and $V = [\gamma_c(F_n^p), F_n]$. Suppose ψ is tame; it induces the automorphism φ of $F_n/\gamma_c(F_n^p)$ which is therefore tame, a contradiction. ∎

The case $c = 2$ which is not covered by Theorem 0.3 is interesting in view of the fact that the group $F_2/\gamma_2(R)$ has only tame automorphisms provided either $R \leq F_2{}'$ and the group ring $\mathbb{Z}(F_2/R)$ is a domain, or the group $G = F_2/R$ is not cyclic, G/G' has at least one infinite cyclic factor, and the group ring $\mathbb{Z}G$ is an Ore domain which has only trivial units (see [BFM]). Theorem 0.6 below seems to be a nice complement to the cited result.

THEOREM 0.6: *Let G be any group with the following property: The group ring $\mathbb{Z}(G)$ has a non-trivial unit such that its image in the group ring $\mathbb{Z}(G/G')$ is also a non-trivial unit. Then there is a presentation $G = F/R$ with F a free group, such that the group F/R' has non-tame automorphisms.*

Using the characterization of abelian groups, the group rings of which have non-trivial units (see [Se]) we obtain the following:

COROLLARY 0.7: *Let A be an abelian group with torsion elements of order 5 or greater than 6. Then A has a presentation F/R such that the group F/R' has non-tame automorphisms.*

To conclude this section, we note that one and the same group might have two different presentations F_1/R_1 and F_2/R_2 such that one of them has non-tame automorphisms while another does not (see [LMR]).

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1. Preliminaries

In this section we will review some results from [LM] which will allow us to recognize non-tame automorphisms. For proofs and further information see [LM].

Throughout the paper G denotes a finitely generated group with generators $x = \{x_1, \ldots, x_n\}$. Let $y = \{y_1, \ldots, y_n\}$ be another generating system for G of the same cardinality. Let $F(X)$ and $F(Y)$ denote the free groups on $X = \{X_1, \ldots, X_n\}$ and $Y = \{Y_1, \ldots, Y_n\}$ respectively. Denote by β_x and β_y the epimorphisms $F(X) \to G$, $F(Y) \to G$ given by $X_i \to x_i$ and $Y_i \to y_i$.

Definition 1.1: The generating systems x, y are **Nielsen equivalent** if there is an isomorphism α: $F(X) \to F(Y)$ so that $\beta_y \cdot \alpha = \beta_x$.

Note that if φ: $G \to G$ is an automorphism it is clear that if φ is tame then the generating systems $x = \{x_1, \ldots, x_n\}$ and $\varphi(x) = \{\varphi(x_1), \ldots, \varphi(x_n)\}$ are Nielsen equivalent.

Let $d_i = \partial/\partial X_i$: $\mathbf{Z}F \to \mathbf{Z}F$, $1 \le i \le n$, denote the i-th Fox derivation of the integral group ring $\mathbf{Z}F(X)$ (see [F]). It is a $\mathbf{Z}$ linear map with the following properties:

(1) $d_i(X_j) = \delta_{ij}$;

(2) $d_i(nu + mv) = nd_i(u) + md_i(v)$; for u, $v \in F(X)$ and m, $n \in \mathbf{Z}$;

(3) $d_i(uv) = d_i(u) + ud_i(v)$.

The following Theorem II of [LM] gives us a tool to distinguish Nielsen inequivalent generating systems of minimal cardinality.

THEOREM II: *Let G be presented by*

$$G = < x_1, \dots, x_n \| R_1, R_2, \dots >,$$

and let a second generating system $y_1 = w_1(x_1, \dots, x_n), \dots, y_n = w_n(x_1, \dots, x_n)$ be given as words in the x_i. For any word $w = w(x_1, \dots, x_n) \in G$ let W denote the corresponding word $W = w(X_1, \dots, X_n) \in F(X)$. Let $\partial w/\partial x_i \in \mathbb{Z}G$ denote the image of the Fox derivative $\partial W/\partial X_i$ under the map β_x: $\mathbb{Z}F(X) \to \mathbb{Z}G$, $X_i \to x_i$.

Let A be a commutative ring with $1 \in A$, and let ρ: $\mathbb{Z}G \to M_m(A)$, $\rho(1) = 1$, be a ring homomorphism so that $\rho(\beta_x(\partial R_k/\partial X_i)) = 0$ for all R_k and X_i. If the determinant of the $(mn \times mn)$-matrix $\rho((\partial w_j/\partial x_i)_{j,i})$ is not contained in the subgroup of A^* generated by the determinants of $\rho(\pm x_1), \dots, \rho(\pm x_n)$, then $x_1, \dots, x_n$ and $y_1, \dots, y_n$ are Nielsen inequivalent generating systems of G.

We will use the notation of [LM] and denote the two-sided ideal $(\beta_x(\partial R/\partial X_i) \| R \in \ker\beta_x)$ of $\mathbb{Z}G$ by I_x. We furthermore remark that Proposition 2.5 of [LM] shows that representations into matrix rings as above do in fact exist.

If $R \subseteq F$ is a normal subgroup of the group F, we have a natural homomorphism ϵ_R: $\mathbb{Z}F \to \mathbb{Z}(F/R)$. We denote $\ker\epsilon_R$ by Δ_R. If $R = F$ we just have the augmentation ideal Δ_F of the group ring $\mathbb{Z}F$.

LEMMA 1.2:

(a) *Let J be an arbitrary ideal of the group ring $\mathbb{Z}F$ and let $u \in \Delta_F$. Then $u \in J\Delta_F$ if and only if $d_i(u) \in J$ for each i, $1 \le i \le n$;*

(b) *Let $y_1, y_2 \in F$ and $[y_1, y_2] = y_1^{-1}y_2^{-1}y_1y_2$. Then:*

$$d_i([y_1, y_2]) = y_1^{-1}y_2^{-1}(1 - y_2)d_i(y_1) + y_1^{-1}y_2^{-1}(y_1 - 1)d_i(y_2).$$

For proofs see [F].

The next lemma describes the group $\gamma_c(R)$ as a subgroup of the free group F determined by some ideal of the group ring $\mathbb{Z}F$ in two ways. The first assertion is usually attributed to Magnus although in the form given here it was proved in [CFL] for the first time. The second assertion is due to Gruenberg (see [Gr] §4.1).

LEMMA 1.3:
(a) $\gamma_c(R) = ({\Delta_R}^c + 1) \cap F$, $c \geq 1$;
(b) $\gamma_c(R) = ({\Delta_R}^{c-1}\Delta_F + 1) \cap F$, $c \geq 2$.

We now illustrate our method by presenting non-tame automorphisms for free nilpotent groups.

Example 1.4: Let $G = F/\gamma_{c+1}(F)$ be a free nilpotent group of an arbitrary class $c \geq 3$ and arbitrary rank $n \geq 2$. Set $\varphi(X_1) = Y_1 = X_1[X_1, X_2, X_1]$; $\varphi(X_i) = Y_i = X_i$ for $i \geq 2$. It is easy to check that φ defines an automorphism of the group G.

We check that φ defines a non-tame automorphism of G. It follows from Lemmas 1.2 and 1.3 that for any $g \in \gamma_{c+1}(F)$ and any i, $1 \leq i \leq n$, $d_i(g) \in {\Delta_F}^c$; hence $d_i(g) \in {\Delta_F}^3$. By sending X_1, $X_2 \to t$ and $X_i \to 1$, $i > 2$, we obtain a homomorphism $\mathbb{Z}F \to \mathbb{Z}[t, t^{-1}]$. The ideal I_x is mapped to 0 in $\mathbb{Z}[t, t^{-1}]/(\operatorname{Im} {\Delta_F}^3)$ thus there is an induced homomorphism on the quotient rings $\mathbb{Z}G \to \mathbb{Z}[t, t^{-1}]/(\operatorname{Im} {\Delta_F}^3)$ (see [LM]).

Consider the image $J\varphi'$ of the Jacobian matrix $J\varphi = (\partial\varphi(x_i)/\partial x_j)$ in the commutative ring $\mathbb{Z}[t, t^{-1}]$. It is immediate that det $J\varphi' = d_1(y_1)$. Using Lemma 1.2 (b) we calculate:

$$d_1(y_1) = 1 + x_1(1 - x_1)(1 - x_2) + [x_1, x_2] - 1 \quad (\bmod {\Delta_F}^3).$$

This yields that the determinant $J\varphi'$ is equal to

$$1 + t(t-1)^2 \bmod (\operatorname{Im}({\Delta_F}^3)). \tag{1}$$

The determinant $J\varphi'$ is not a trivial unit of $\mathbb{Z}[t, t^{-1}]$ modulo the image of ${\Delta_F}^3$. Indeed, the image of ${\Delta_F}^3$ consists of polynomials of the form $P(t, t^{-1})(t-1)^3$. Were the expression (1) a trivial unit of $\mathbb{Z}[t, t^{-1}]$ modulo the image of ${\Delta_F}^3$, we would have:

$$t(t-1)^2 = \pm t^k - 1 + P(t, t^{-1})(t-1)^3$$

for some polynomial $P(t, t^{-1})$ and some integer k. This is a contradiction because the polynomial on the left-hand side has 1 as a root of multiplicity two while the polynomial on the right-hand side has 1 as a root of multiplicity ≤ 1 or ≥ 3. The proof is now completed by applying Theorem II to the case $m = 1$ and $A^* = (\mathbb{Z}[t, t^{-1}]/\operatorname{Im}({\Delta_F}^3))^*$.

2. Central Extensions of Burnside Groups

It is natural, in light of Remark 0.2, to ask whether the central extensions of free Burnside groups $G = F_n/[F_n^p, F_n]$ have non-tame automorphisms. As the abelianization of G is Z^n it is impossible to use the idea of the example in Remark 0.2. In this section we will answer this question in the affirmative. The device with which we will determine this is Theorem II of [LM] which is an application of the $\mathcal{N}$-torsion invariant.

LEMMA 2.1: *Assume $n \geq 3$. The map $\varphi_n \colon G \to G$ defined by*

$$\varphi_n(x_1) = x_1^p x_2 x_3^{-(p+1)}, \quad \varphi_n(x_2) = x_2^p x_3, \quad \varphi_n(x_3) = x_1^{-(p-1)} x_3^p,$$
$$\varphi_n(x_i) = x_i \quad \text{for } i > 3$$

is an automorphism.

Proof: As $[F_n^p, F_n]$ is a fully invariant subgroup of F_n it is clear that φ_n is a homomorphism. To check that it is an automorphism it is sufficient to produce an inverse map. Consider the homomorphism $\Psi_n \colon G \to G$ defined by :

$$\Psi_n(x_1) = x_1^{p^2} x_2^{-p} x_3^{p^2+p+1} C_1, \quad \Psi_n(x_2) = x_1^{-(p-1)} x_2 x_3^{-p} C_2,$$
$$\Psi_n(x_3) = x_1^{p(p-1)} x_2^{-(p-1)} x_3^{p2} C_3, \quad \Psi_n(x_i) = x_i, \qquad i > 3,$$

where

$$C_2 = x_1^p x_2^p (x_1 x_2)^{-p}, \quad C_3 = x_1^p x_2^p (x_1 x_2 C_2)^{-p}, \quad C_1 = C_3 C_2^{-1}.$$

A computation shows that $\varphi_n(\Psi_n(x_j)) = x_j$ for each j. ∎

LEMMA 2.2: *Let $p \geq 4$ be even. Then the automorphism φ_n is not tame.*

Proof: Consider the Jacobian matrix $J\varphi = (\partial \varphi_n(x_i)/\partial x_j)$, as in Theorem II. It is a diagonal block matrix over $\mathbb{Z}G$ with a $(n-3) \times (n-3)$-identity block in the lower right corner and a (3×3)-block in the upper left corner of the form:

$$\begin{bmatrix} 1+x_1+\cdots+x_1^{p-1} & x_1^p & -x_1^p x_2 x_3^{-(p-1)}(1+x_3+\cdots+x_3^p) \\ 0 & 1+x_2+\cdots+x_2^{p-1} & x_2^p \\ -x_1^{-(p-1)}(1+x_1+\cdots+x_1^{p-2}) & 0 & x_1^{-(p-1)}(1+x_3+\cdots+x_3^{p-1}) \end{bmatrix}.$$

We obtain a map $\mathbb{Z}[F_n/[F_n^p, F_n]] \to \mathbb{Z}$ by sending $x_1, x_3 \to -1$, $x_2, x_i \to 1$, $i \geq 4$, and extending linearly over $\mathbb{Z}[F_n/[F_n^p, F_n]]$. The image of the Jacobian matrix $J\varphi$ is the $(n \times n)$-block matrix with an $(n-3) \times (n-3)$-identity block and a (3×3)-block

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & p & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

It has determinant $1-p$.

The group $G = F_n/[F_n^p, F_n]$ has a presentation:

$$G = < x_1, \dots, x_n \| [g^p, f],\ g, f \in F_n > .$$

We compute the image of $\partial[g^p, f]/\partial X_i$ over $\mathbb{Z}G$ denoted by $\partial[g^p, f]/\partial x_i$ as in Theorem II:

$$\begin{aligned} \partial[g^p, f]/\partial x_i = & - g^p(1-f^{-1})(1+g+\cdots+g^{p-1})\partial g/\partial x_i \\ & + g^{-p}f^{-1}(1-g)(1+g+\cdots+g^{p-1})\partial f/\partial x_i \end{aligned}$$

The images in $\mathbb{Z}$ of each g, $f \in F$ are ± 1. As p is even, if $g \to \pm 1$ the second summand is 0. If $g \to 1$ and $f \to -1$ the first summand is $-2p$ and if $f \to 1$ the first summand is 0. In any case the image of $\partial[g^p, f]/\partial x_i$ is contained in the ideal $(2p) \subset \mathbb{Z}$ for each f, g and $x_i \in \{x_1, ..., x_n\}$. Thus we have a representation ρ: $\mathbb{Z}[F_n/[F_n^p, F_n]] \to \mathbb{Z}/(2p)$ in which $\rho(\partial[g^p, f]/\partial x_i) = 0$. We are now in position to apply Theorem II with $m = 1$ and $A^* = (\mathbb{Z}/(2p))^*$. The trivial units $\rho(x_i)$ are ± 1. It is clear that unless $p = 2$ (p is even) $p-1 \not\equiv \pm 1 \bmod 2p$. Thus, by Theorem II, the generating systems $x = \{x_1, ..., x_n\}$ and $\varphi(x) = \{\varphi(x_1), ..., \varphi(x_n)\}$ are not Nielsen equivalent and hence φ is not tame. ∎

LEMMA 2.3: *Assume $n \geq 4$. The map φ_n: $G \to G$ defined by*

$$\varphi_n(x_1) = x_2^p x_3^{(p+1)} x_4^p, \quad \varphi_n(x_2) = x_2 x_3^p x_4^2, \quad \varphi_n(x_3) = x_1^p x_3^{p+1} x_2^{p+1} x_4^{p+1},$$

$$\varphi_n(x_4) = x_1^{p-1} x_2^p x_3^p x_4^p, \quad \varphi_n(x_i) = x_i \quad \textit{for } i > 4,$$

is an automorphism.

Proof: As $[F_n^p, F_n]$ is a fully invariant subgroup of F_n it is clear that φ_n is a homomorphism. To check that it is an automorphism it is sufficient to produce an inverse map. The homomorphism φ_n induces an isomorphism $\varphi_n^{\mathrm{ab}} \in \mathrm{GL}(n, \mathbb{Z})$ on the abelianized group $G/G' = \mathbb{Z}^n$. The matrix φ_n^{ab} is an $(n \times n)$-block matrix which is an $(n-4)\times(n-4)$-identity block on the lower right corner and a (4×4)-matrix in the upper left corner of the form

$$\begin{bmatrix} 0 & p & p+1 & p \\ 0 & 1 & p & 2 \\ p & p+1 & p+1 & p+1 \\ p-1 & p & p & p \end{bmatrix}$$

The inverse to φ_n^{ab} is an $(n\times n)$-block matrix $(\varphi_n^{ab})^{-1}$ which is an $(n-4)\times(n-4)$-identity block on the lower right corner and a (4×4)-matrix in the upper left corner of the form:

$$\begin{bmatrix} 0 & 0 & p & -(p+1) \\ p-2 & -1 & p(p-1)(2(p+1)-p^2) & p(2(p+1)-p^2) \\ 1 & 0 & p(p-1) & -p^2 \\ -p+1 & 1 & -p^3+2p^2+1 & p^3-p^2-p \end{bmatrix}$$

Hence if φ_n has an inverse Ψ_n then it must induce the map $(\varphi_n^{\mathrm{ab}})^{-1}$ on $\mathbb{Z}^n$. Thus Ψ_n must be of the form:

$$\begin{aligned} &\Psi_n(x_1) = x_3^p x_4^{-(p+1)} C_1, \quad \Psi_n(x_2)^{p-2} x_2^{-1} x_3^{p^3-3p^2+2} x_4^{-p^3+2p^2+2p} C_2, \\ &\Psi_n(x_3) = x_1 x_3^{p(p-1)} x_4^{-p^2} C_3, \quad \Psi_n(x_4) = x_1^{-p+1} x_2 x_3^{-p^3+2p^2-1} x_4^{p^3-p^2-p} C_4, \\ &\Psi_n(x_i) = x_i C_i, \qquad i>4, \end{aligned}$$

where C_1, C_2, C_3, C_4, C_i, $5\le i\le n$, are unknown elements in the commutator subgroup of G.

The compositions $\Psi_n(\varphi_n(x_i)) = x_i$ give us equations in C_1, C_2, C_3, C_4, C_i. If these equations have solutions in the commutator subgroup of G then Ψ_n as above is an inverse to φ_n. The equations are:

$$x_i C_i = x_i \quad \text{for } 5\le i\le n, \quad \text{and} \tag{0}$$

$$x_3^{-p}(x_1^{-2}x_2^{-1}x_3^2C_2)^p(x_1C_3)^{p+1}(x_1x_2x_3^{-1}C_4)^p = x_1, \tag{1}$$

$$x_1^{-p-2}x_2^{-1}x_3^2C_2(x_1C_3)^p(x_1x_2x_3^{-1}C_4)^2 = x_2, \tag{2}$$

$$x_3^{-p}x_4^p(x_4^{-1}C_1)^p(x_1C_3)^{p+1}(x_1^{-2}x_2^{-1}x_3^2C_2)^{p+1}(x_1x_2x_3^{-1}C_4)^{p+1} = x_3, \tag{3}$$

$$(4) \qquad x_3^{-p}x_4^p(x_4^{-1}C_1)^{p-1}(x_1^{-2}x_2^{-1}x_3^2C_2)^p(x_1C_3)^p(x_1x_2x_3^{-1}C_4)^p = x_4.$$

So we need to present a solution to the equations for C_1, C_2, C_3, C_4, C_i. Set $C_i = 1$, for $5 \leq i \leq n$. Equations (1) and (4) can be rewritten :

$$(1^*) \qquad x_3^{-p}(x_1^{-2}x_2^{-1}x_3^2C_2)^p(x_1C_3)^p(x_1x_2x_3^{-1}C_4)^p = C_3^{-1},$$

$$(4^*) \qquad x_3^{-p}x_4^p(x_4^{-1}C_1)^p(x_1^{-2}x_2^{-1}x_3^2C_2)^p(x_1C_3)^p(x_1x_2x_3^{-1}C_4)^p = C_1.$$

Hence, as C_1 and C_3 are p-powers, we can assume without loss of generality that they are in the center of G. Set $A = x_1^{-2}x_2^{-1}x_3^2C_2$ and $B = x_1x_2x_3^{-1}C_4$. We can then rewrite the equations (1)–(4) as:

$$(1') \qquad x_1^p x_3^{-p}C_3^{p+1}A^pB^p = 1,$$

$$(2') \qquad x_2^{-1}C_3^pAB^2 = 1,$$

$$(3') \qquad C_1^pC_3^{p+1}x_3^{-p-1}x_1^{p+1}A^{p+1}B^{p+1} = 1,$$

$$(4') \qquad x_3^{-p}x_1^pC_1^{p-1}C_3^pA^pB^p = 1.$$

From (3′) and (4′) we obtain

$$(5') \qquad C_1C_3x_3^{-1}x_1AB = 1.$$

From (1′) and (3′) we obtain

$$(6') \qquad C_1^px_3^{-1}x_1AB = 1.$$

From (5′) and (6′) we obtain

$$(7') \qquad C_3 = C_1^{p-1}.$$

From (6′) we get $AB = C_1^{-p}x_1^{-1}x_3$ and substituting this in (2′) yields:

$$(8') \qquad B = C_1^pC_3^{-p}x_3^{-1}x_1x_2 \quad \text{and} \quad A = C_3^pC_1^{-2p}x_1^{-1}x_3x_2^{-1}x_1^{-1}x_3.$$

Thus, using ($7'$), we can solve for C_2 and C_4 in terms of C_1 and $\{x_1, ..., x_4\}$. Substitute A,B and ($7'$) back in ($1'$) and set $A' = x_1^{-1}x_3x_2^{-1}x_1^{-1}x_3$, $B' = x_3^{-1}x_1x_2$. We get:

$$x_1^p x_3^{-p} C_1^{p^2-1} C_1^{p^2(p-1)} C_1^{-2p^2} (A')^p C_1^{p^2} C_1^{-p^2(p-1)} (B')^p = 1.$$

As C_1 is in the center we get $C_1^{-1}x_1^p x_3^{-p}(A')^p(B')^p = 1$, i.e.,

$$C_1 = x_1^p x_3^{-p}(x_1^{-1}x_3x_2^{-1}x_1^{-1}x_3)^p(x_3^{-1}x_1x_2)^p.$$

It is immediate to check that C_1 and hence C_3, are in the commutaтor subgroup of G. The fact that C_2, C_4 are also in the commutator subgroup folows from ($8'$) and the definitions of A, B. Therefore we have solutions to the equations (0)–(4), and φ_n thus defined is an automorphism. ∎

LEMMA 2.4: *Let p be odd. Then the automorphism φ_n is not tame.*

Proof: Consider the Jacobian matrix $J\varphi$. It is a diagonal block matrix over ZG with a $(n-4)\times(n-4)$-identity block on the lower right corner and a (4×4)-block on the upper left corner of the form (which, unfortunately, is too wide to fit in one line):

$$\left[\begin{matrix} 0 & 1+x_2+\cdots+x_2^{p-1} \\ 0 & 1 \\ 1+x_1+\cdots+x_1^{p-1} & x_1^p x_3^{p+1}(1+x_2+\cdots+x_2^p) \\ 1+x_1+\cdots+x_1^{p-2} & x_1^{p-1}(1+x_2+\cdots+x_2^{p-1}) \end{matrix}\right.$$

$$\left.\begin{matrix} x_2^p(1+x_3+\cdots+x_3^p) & x_2^p x_3^{p+1}(1+x_4+\cdots+x_4^{p-1}) \\ x_2(1+x_3+\cdots+x_3^{p-1}) & x_2x_3^p(1+x_4) \\ x_1^p(1+x_3+\cdots+x_3^p) & x_1^p x_3^{p+1}x_2^{p+1}(1+x_4+\cdots+x_4^p) \\ x_1^{p-1}x_2^p(1+x_3+\cdots+x_3^{p-1}) & x_1^{p-1}x_2^p x_3^p(1+x_4+\cdots+x_4^{p-1}) \end{matrix}\right]$$

We get a map $\mathbb{Z}[F_n/[F_n^p, F_n]] \to \mathbb{Z}[\xi]$, $\xi = e^{2\pi i/p}$, by sending x_1, $x_4 \to 1$ and x_2, $x_3 \to \xi$. The image of the above matrix is the following (4×4)-matrix over $\mathbb{Z}[\xi]$:

$$\begin{bmatrix} 0 & 0 & 1 & \xi p \\ 0 & 1 & 0 & 2\xi \\ p & \xi & 1 & \xi^2(p+1) \\ p-1 & 0 & 0 & p \end{bmatrix}$$

with determinant $-p^2-(p-1)(-\xi^2 p+\xi p+\xi^2) \subset Z[\xi]$. As in Lemma 2.2 we compute:

$$\begin{aligned}\partial[g^p, f]/\partial x_i = &- g^p(1-f^{-1})(1+g+\ldots+g^{p-1})\partial g/\partial x_i \\ &+ g^{-p}f^{-1}(1-g)(1+g+\ldots+^{p-1})\partial f/\partial x_i\end{aligned}$$

If $g \to \xi^m$ then $\partial[g^p, f]/\partial x_i \to 0$ and if $g \to 1$, $f \to \xi^m$ then the second summand is mapped to 0 and the first summand is mapped to $-(1-\xi^{-m})p$. In both cases the image of $\partial[g^p, f]/\partial x_i$ is contained in the ideal generated by $\{((1-\xi^{-m})p) \| m \in \mathbb{Z}\} \subset ((1-\xi)p) \subset \mathbb{Z}[\xi]$.

Thus we have obtained a representation $\rho: \mathbb{Z}[F_n/[F_n^p, F_n]] \to \mathbb{Z}[\xi]/((1-\xi)p)$, $\xi = e^{2\pi i/p}$, such that $\rho(\partial[g^p, f]/\partial x_i) = 0$. We can now apply Theorem II for the case $m = 1$ and $A^* = (\mathbb{Z}[\xi]/(1-\xi)p)^*$. We can conclude that the generating systems $x = \{x_1, ..., x_n\}$ and $\varphi(x) = \{\varphi(x_1), ..., \varphi(x_n)\}$ are not Nielsen equivalent and hence φ is not tame if we can show that

$$-p^2 - (p-1)(-\xi^2 p + \xi p + \xi^2) \neq \pm\xi^m \bmod ((1-\xi)p) \subset \mathbb{Z}[\xi].$$

LEMMA 2.5: *Let $p \in \mathbb{Z}$ be odd where $\xi = e^{2\pi i/p}$, then:*

$$-p^2 - (p-1)(-\xi^2 p + \xi p + \xi^2) \neq \pm\xi^m \bmod ((1-\xi)p) \text{ in } \mathbb{Z}[\xi].$$

Proof: Note that $p = \xi^m p$ for each $m \neq p$ in the ring $R = \mathbb{Z}[\xi]/((1-\xi)p)$. Hence we have

$$0 = (1 + \xi + \xi^2 + \cdots + \xi^{p-1}) = p + p + \cdots + p = p^2$$

This implies that $-p^2 - (p-1)(-\xi^2 p + \xi p + \xi^2) = \xi^2 - p$ in R. Notice that in R the element $\xi^2 - p$ is a unit as $(\xi^2 - p)(\xi^{p-2} + p) = \xi^p + \xi^2 p - \xi^{p-2}p + p^2 = 1$. When multiplied by the trivial unit ξ^{p-2} the image of the determinant $\xi^2 - p$ becomes $(1-p)$. Hence in order to prove the Lemma we need to show that $(1-p) \neq \pm\xi^m \bmod ((1-\xi)p)$ in $\mathbb{Z}[\xi]$.

CASE (A): *If $(1-p) = -\xi^m \bmod ((1-\xi)p)$ then $(1-p) = -\xi^m + (1-\xi)pr$ for some $r \in \mathbb{Z}[\xi]$.* We multiply both sides by p to get $p - p^2 = -\xi^m p + (1-\xi)pr'$ or $2p \in ((1-\xi)p)$ in $\mathbb{Z}[\xi]$. But then $2 \in (1-\xi)$. Recall that $1-\xi$ is not invertible in $\mathbb{Z}[\xi]$ and $2 \neq 0$ in $\mathbb{Z}/p\mathbb{Z} = \mathbb{Z}[\xi]/(1-\xi)$.

CASE (B): *If $(1-p) = \xi^m \bmod ((1-\xi)p)$ then $(1-p) = \xi^m + (1-\xi)pr$ for some $r \in \mathbb{Z}[\xi]$. But then $1 - \xi^m = (1-\xi)pr + p$.* This is a contradiction as the coefficients of the polynomial on the left hand side are not divisible by p but the coefficients of the polynomial on the right hand side are, and p is not a unit in $\mathbb{Z}[\xi]$. ∎

This concludes the proof of Lemma 2.4. ∎

Proof of Theorem 0.1: Theorem 0.1 is an immediate consequence of Lemmas 2.1, 2.2, 2.3 and 2.4. ∎

3. Free Nilpotent Extensions of Burnside Groups

In this section we prove Theorem 0.3 and Theorem 0.6.

Proof of Theorem 0.3: Let $B(n,p)$ be the free Burnside group and $G_c = F_n/\gamma_c(F_n^p)$ its nilpotent extension. Define a map φ: $F_n \to F_n$ by $\varphi(X_1) = Y_1 = X_1[X_1^p, X_2^p, X_1]$; $\varphi(X_i) = Y_i = X_i$ for $i \geq 2$. As φ induces the identity map modulo $\gamma_2(R) = R'$, $R = F_n^p$, we can apply Lemma 3.1 of [BG] and conclude that φ induces an automorphism φ of the group G_c.

We check now that φ defines a non-tame automorphism of G_c. It follows from Lemmas 1.2 and 1.3 that for any $g \in \gamma_c(R)$ and any i, $1 \leq i \leq n$, $d_i(g) \in \Delta_R^{c-1}$ hence $d_i(g) \in \Delta_R^2$. By sending X_1, $X_2 \to t$ and $X_i \to 1$, $i > 2$, we obtain a homomorphism $\mathbb{Z}F_n \to \mathbb{Z}[t,t^{-1}]$ and the ideal I_x is mapped to 0 in $\mathbb{Z}[t,t^{-1}]/(\operatorname{Im}\Delta_R^2)$. As before there is an induced homomorphism on the quotient rings $\mathbb{Z}G \to \mathbb{Z}[t,t^{-1}]/(\operatorname{Im}\Delta_F{}^3)$ (see [LM]).

Consider now the image $J\varphi'$ of the Jacobian matrix $J\varphi = (\partial\varphi(x_i)/\partial x_j)$ in the commutative ring $\mathbb{Z}[t,t^{-1}]$. It is immediate that $\det J\varphi' = \operatorname{Im} d_1(y_1)$. Using Lemma 1.2 (b) we calculate:

$$d_1(y_1) = 1 + x_1(1-x_1)(1-x_2^p)(1+x_1+\cdots+x_1^{p-1})(\bmod \Delta_R^2).$$

This yields that the determinant $J\varphi'$ is equal to

$$1 + t(t-1)(t^p-1)(1+t+\cdots+t^{p-1}) \bmod (\operatorname{Im}(\Delta_R^2)). \tag{1}$$

The determinant $J\varphi'$ is not a trivial unit of $\mathbb{Z}[t,t^{-1}]$ modulo the image of Δ_R^2. Indeed, the image of Δ_R^2 consists of polynomials of the form $P(t,t^{-1})(t^p-1)^2$. Were the expression (1) a trivial unit of $\mathbb{Z}[t,t^{-1}]$ modulo the image of Δ_R^2, we would have:

$$t(t-1)(t^p-1)(1+t+\cdots+t^{p-1}) = \pm t^k - 1 + P(t,t^{-1})(t^p-1)^2$$

for some polynomial $P(t,t^{-1})$ and some integer k. This is a contradiction because the polynomial on the left-hand side has 1 as a root of multiplicity two while the polynomial on the right-hand side has 1 as a root of multiplicity ≤ 1. The proof is now completed by applying Theorem II to the case $m = 1$ and $A^* = (\mathbb{Z}[t,t^{-1}]/\operatorname{Im}(\Delta_R^2))^*$. ∎

Proof of Theorem 0.6: Let $x_1,\ldots,x_n$ be some minimal set of generators for the group G. Then there is a free group of rank n, $F_n = F(X_1,\ldots,X_n)$ such that G has a presentation of the form F_n/S. Consider the free group $F_{n+1} = F(X_1, ..., X_n, X_{n+1})$ of rank $n+1$ and its normal subgroup R generated by S and X_{n+1} (we consider F_n to be naturally embedded in F_{n+1}). Clearly $G = F_{n+1}/R$. We now present a non-tame automorphism of the group F_{n+1}/R'.

There is a well-known action of the group ring $\mathbb{Z}(F/R)$, $F = F_{n+1}$ on the abelian group R/R' giving rise to the notion of relation module of F/R (see [LS] p. 100). For $h \in F/R$, and $r \in R/R'$, there is a map $(h,r) \to hrh^{-1} \in R/R'$. For an arbitrary $v \in \mathbb{Z}(F/R)$ extend this map $\mathbb{Z}$-linearly. For any $v \in \mathbb{Z}(F/R)$, r^v will denote the result of the action of v on $r \bmod R'$. It is straight-forward to see that $d_i(r^v) = vd_i(r)(\bmod \Delta_R)$, $1 \le i \le n$.

As X_{n+1} is an element of the group R, this action is defined on X_{n+1} modulo R'. Choose a non-trivial unit $U \in \mathbb{Z}(G)$ such that its natural image $u \in Z(G/G')$ is also a non-trivial unit. Define a map φ: $F/R' \to F/R'$ by

$$\varphi(X_{n+1}) = Y_{n+1} = X_{n+1}^U; \quad \varphi(X_i) = Y_i = X_i \quad \text{for } 1 \le i \le n.$$

It is easy to see that φ induces a homomorphism of the group F/R', and it has an inverse defined by $Y_i \to X_i$ for $1 \le i \le n$, $Y_{n+1} \to X_{n+1}^{U^{-1}}$.

Consider the homomorphism $\mathbb{Z}(F/R) \to \mathbb{Z}(G/G')$. It is immediate that the image of det $J\varphi$ is equal to $u = \mathrm{Im}(d_{n+1}(Y_{n+1}))$ which is not a trivial unit of $Z(G/G')$. As any derivative of any element of R' is contained in Δ_R the image of I_x is contained in $(\mathrm{Im}\,\Delta_R)$. The proof is now completed by applying Theorem II to the case $m = 1$ and $A^* = (Z(G/G')/\mathrm{Im}(\Delta_R))^*$. ∎

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